Today’s Lecture

• Subband Coding Motivation
• Brief review of QMF analysis
• Designs:
  – Eliminating aliasing
  – Approximating perfect reconstruction:
    • IIR filtering
    • Johnston’s technique
  – Actual PR: Conjugate Quadrature Filters (CQF)
    • We won’t cover in detail.
• Next lecture: application to progressive image compression
Subband Coding

Want to divide signal into 4 bands and quantize the signal independently in each band.
Subband Coding

Above is no longer a filter bank. It is a “filter tree.”

However, the basic building block of this filter tree is the Quadrature Mirror Filter (QMF), which is a filter bank.

In the above, the quantizer is a digital-to-digital “requantizer” used to achieve signal compression.
Subband Coding

Specs for filters:

One possibility:

Ideal filters

These two filters form a QMF pair.
Subband Coding
Quadrature Mirror Filters:
Design $H_0(z)$, $H_1(z)$, $G_0(z)$, and $G_1(z)$;

We ignore signal quantization of subbands. That is a “source coding” problem. Ideally want perfect reconstruction.
QMF Analysis Recap

\[ \hat{X}(z) = \frac{1}{2} \left[ H_0(z) G_0(z) + H_1(z) G_1(z) \right] X(z) \]

\[ + \frac{1}{2} \left[ H_0(-z) G_0(z) + H_1(-z) G_1(z) \right] X(-z) \]

(derived in previous lecture)
QMF Analysis Recap

\[ \hat{X}(z) = \frac{1}{2} \left[ H_0(z) G_0(z) + H_1(z) G_1(z) \right] X(z) \]
\[ + \frac{1}{2} \left[ H_0(-z) G_0(z) + H_1(-z) G_1(z) \right] X(-\frac{1}{2}) \]

aliasing term

To cancel aliasing in \( \hat{X}(z) \), we can set

(by inspection)

\[ G_0(z) = H_1(-z) \]
\[ G_1(z) = -H_0(-z) \]
QMF Analysis Recap

\[ \hat{X}(z) = \frac{1}{2} \left[ H_0(z) G_0(z) + H_1(z) G_1(z) \right] X(z) \]

\[ = \frac{1}{2} \left[ H_0(z) H_1(-z) - H_1(z) H_0(-z) \right] X(z) \]

\[ \equiv T(z) \]

If \( T(z) \) is ALL-PASS, then NO Amplitude distortion.

If \( T(z) \) is LINEAR-PHASE, then NO Phase distortion.

Pick: \( H_1(z) = H_0(-z) \)

Alias Free \( \Rightarrow \) \( G_0(z) = H_0(z) \), \( G_1(z) = -H_1(z) \)

Only 1 filter must be designed.
QMF Analysis Recap

Means

Perfect Reconstruction QMF bank:

\[ H_1(z) = H_0(-z) \]

Alias free \[ \Rightarrow G_0(z) = H_0(z), \quad G_1(z) = -H_1(z) \]

Only 1 filter must be designed.

The other 3 filters are derived from \( H_0(z) \).
Alias Free QMFs with “Simple” PR condition

Then \( \hat{X}(z) = T(z) \ast X(z) \)

where \( T(z) = H_0(z)G_0(z) + H_1(z)G_1(z) \), \( G_0(z) = H_0(z) \), \( G_1(z) = -H_1(z) \)

\[ T(z) = \frac{1}{2} \left[ H_0^2(z) - H_1^2(z) \right] \]

\[ = \frac{1}{2} \left[ H_0^2(z) - H_0^2(-z) \right] \]

Effect of choice of \( H_1(z) = H_0(-z) \):

Let \( H_0(z) = E_0(z^2) + z^{-1}E_1(z^2) \)

\( H_1(z) = H_0(-z) \)

\[ = E_0(\bar{z}^2) - z^{-1}E_1(z^2) \]

(Polyphase representations)
Alias Free QMFs with “Simple” PR condition

\[ T(z) = \frac{1}{2} \left[ H_0^2(z) - H_0^2(-z) \right] \]

\[ = \frac{1}{2} \left\{ \left[ E_0(z^2) + z^{-1} E_1(z^2) \right]^2 - \left[ E_0(z^2) - z^{-1} E_1(z^2) \right]^2 \right\} \]

\[ = 2z^{-1}E_0(z^2)E_1(z^2) \]

=> If \( H_0(z) \) is FIR, all other filters are FIR. \((E_0(z^2), E_1(z^2))\)

Each FIR filter must be a pure delay for perfect reconstruction, i.e., no amplitude or phase distortion.

=> These are NOT good filters for separation into subbands.

=> Must use IIR filter for \( H_1(z) \)
Alias Free QMFs with “Simple” PR condition

That is,

\[ E_0(\tau) = c_0 z^{-r_0}, \quad E_1(\tau) = c_1 z^{-r_1} \]

(required pure delays)

\[ H_0(\tau) = c_0 z^{-2r_0} + c_1 z^{-(2r_1+1)} \]
\[ H_1(\tau) = c_0 z^{-2r_0} - c_1 z^{-(2r_1+1)} \]

If we choose \( E_1(\tau) = \frac{1}{E_0(\tau)} \), then \( T(\tau) \) becomes a pure delay.

But the filters become IIR.

Unfortunately, IIR filters introduce phase distortion.
QMFs: Alias Free with Approximate PR

Eliminating Phase Distortion
Consider FIR, N odd: \( H_0(z) = \sum_{n=0}^{N} h_0(n)z^{-n} \)

For linear-phase \( H_0(z) \), we require the symmetry \( h_0(n) = h_0(N-n) \) (or, we could have \( h_0(n) = -h_0(N-n) \)).

But we don’t use \( h_0(n) = -h_0(N-n) \) since we cannot get a LPF.

Why? Recall lectures on filter design: for any asymmetric FIR filter as above, the DC response must zero – (Recall Lecture 12)
QMFs: Alias Free with Approximate PR

Why use $N$ odd?

$$T(z) = \frac{1}{2} \left[ H_0^2(z) - H_0^2(-z) \right]^{(-1)^N}$$

$$T(\omega) = \frac{e^{-j\omega N}}{2} \left[ |H_0(\omega)|^2 - e^{+j\pi N} |H_0(\omega-\pi)|^2 \right]$$

$T(\omega) \equiv T(z) \mid_{z=e^{j\omega}}$ normalized frequency in radians/sample

$$H_0(-z) \mid_{z=e^{j\omega}} = H_0(-e^{j\omega}) = H_0(e^{-j(\pi-\omega)})$$

$$= H_0(\omega-\pi) = e^{-j(\omega-\pi)\frac{N}{2}} |H_0(\omega-\pi)|$$

Using $H_1(z) = H_0(-z)$, $|H_1(\omega)| = |H_0(\omega-\pi)|$

linear phase
QMFs: Alias Free with Approximate PR

\[ T(w) = \frac{e^{-j\omega N}}{2} \left[ |H_0(w)|^2 - e^{j\pi N} |H_0(w - \pi)|^2 \right] \]

If \( N \) were even, \( T(w) = 0 \) at \( w = \pi/2 \). Severe Amplitude Distortion!
For \( N \) odd, \( T(w) = \frac{e^{-j\omega N}}{2} \left[ |H_0(w)|^2 + |H_1(w)|^2 \right] \)

**Power Complementarity Condition:**

This term must be constant.

This becomes an approximation problem.
QMFs: Alias Free with Approximate PR: Power Complementarity (PC)

Minimization of residual AMP:

\[ |H_0(w)|^2 + |H_1(w)|^2 \] can only be approx. constant.

Johnston technique: minimize

\[ \phi = \alpha \phi_1 + (1-\alpha) \phi_2 \]

where

\[ \phi_1 = \int_0^\pi |H_0(w)|^2 dw, \quad \phi_2 = \int_0^\pi (1-|H_0(w)|^2 - |H_1(w)|^2)^2 dw \]

Note: Choose \( \omega_s = \frac{\pi}{2} + \epsilon \), \( \epsilon > 0 \), small can replace \( \int_0^\pi \) with \( 2 \int_\frac{\pi}{2}^\pi \)

This uses a numerical optimization technique.

QMFs: Alias Free with Approximate PR: Power Complementarity (PC)

Typical magnitude responses of Johnston filters:
Quadrature Mirror Filters: for PC

Approximation of Power Complementarity (PC) of Johnston filters: (note the exaggerated scale on vertical axis)
Quadrature Mirror Filter (QMF):
Summary of Design Objectives

1. Eliminate *aliasing* caused by the decimation. The decimation creates a shifted version of the signal frequency spectrum centered at \( f = \frac{1}{4} \) (\( \omega = \frac{\pi}{2} \) rad/sample).

2. *Perfect reconstruction (PR)*: no aliasing, no amplitude distortion, and no phase distortion.
   - Aliasing cancellation not difficult
   - If \( H_1(z) = H_0(-z) \), PR comes at the cost of poor analysis filters (defeats purpose).

3. Can forego PR for better analysis filters: either have amplitude or phase distortion.
   - Phase distortion, very small amplitude distortion (IIR filters)
   - No phase distortion, small amplitude distortion (Johnston filters)

4. Get obtain PR with and good FIR analysis filters: Conjugate Quadrature Filters (CQFs). But… the analysis filters themselves are not linear phase.

Details on CQF filter design can be found on rest of slides. There is not enough time to cover this in detail. (This is the state of the art)
Conjugate Quadrature Filters (CQFs)

Want: FIR filters, perfect reconstruction (no aliasing)

Recall: \( \hat{X}(z) = X(z) \left[ \frac{1}{2} \left( H_0(z) H_1(-z) - H_1(z) H_0(-z) \right) \right] \)
\( G_0(z) = H_1(-z) \), \( G_1(z) = -H_0(-z) \)

Rather than setting \( H_1(z) = H_0(-z) \), we instead choose \( H_1(z) = -z^{-N} H_0(-z^{-1}) \) for odd \( N \) (CQF)

We no longer require \( H_0(z) \) to have linear phase.

Substituting this new condition into our distortion expression…
Conjugate Quadrature Filters (CQFs)

Substitute

\[ H_1(z) = -z^{-N} H_0(-z^{-1}) \]

into

\[ \hat{X}(z) = \frac{X(z)}{2} \left[ H_0(z) H_1(-z) - H_1(z) H_0(-z) \right] \]

We require \( H_0(z) H_0(z^{-1}) + H_0(-z) H_0(-z^{-1}) \) to have constant magnitude and linear phase.
Looking at the CQF in the time domain

\[
\text{CQF in time domain: } h_1(n) = 3^{-1} \{ H_1(z) \} \\
\text{NOTE: } h_0(-n) \leftrightarrow H_0(z^{-1}) \quad \text{(time-reversal)} \\
(-1)^n \quad h_0(-n) \leftrightarrow H_0(z^{-1}) \quad \text{(scaling)} \\
(-1)^n \quad h_0(N-n) \leftrightarrow z^{-N} H_0(z^{-1}) \quad \text{(time-shift)} \\
\]

\[
h_1(n) = (-1)^n h_0(N-n) \text{ sign-alternating, time-reversed}
\]

\[
g_0(n) = 3^{-1} \{ H_1(-z) \} = h_0(N-n) \text{ time-reversed}
\]

\[
g_1(n) = 3^{-1} \{ -H_0(-z) \} = (-1)^n h_1(N-n) \text{ time-reversed}
\]
Product Filters

As long as $N > \text{order of } H_0(z)$, all filters are causal if $h_0(n)$ is causal. Only $H_0(z)$ needs to be designed (Smith & Barnwell)

Define "Product Filters" 

\begin{align*}
F_0(z) &= H_0(z) H_0(z^{-1}) \\
F_1(z) &= H_0(-z) H_0(-z^{-1}) \\
&= F_0(-z)
\end{align*}

Let 

$$C(w) = \frac{1}{2} \left[ F_0(w) + F_1(w) \right] e^{-jwN}$$

$C(z)$ is the overall transfer function of filter bank.
Time Domain Product Filters

Ignoring the delay of N samples which can be added at the end,

\[
c(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} C(\omega) e^{j\omega n} d\omega = \delta(n)
\]

\[
\frac{f_0(n)}{2} + \left( -1 \right)^n \frac{f_0(n)}{2} = \delta(n)
\]

\[
\text{or } f_0(n) \left[ 1 + (-1)^n \right] = \delta(n) \text{ is needed.}
\]

in order to have zero phase and constant magnitude response, e.g., perfect reconstruction.
Product Filters (cont.)

Decompose $F_0(w) = V(w) + A$ \( A = \text{constant} \).

Or, equivalently,

$F_0(n) = V(n) + A \delta(n)$, where $V(0) = 0$.

Need $A = 1$.

Therefore, $A$ must be equal to 1 (see previous slide),

and $V(2n) = 0$, $n = 0, \pm 1, \pm 2, \ldots$

Or, equivalently,

$V(w) = -V(\pi - w)$ odd symm.

about $\pi/2$.

$\Rightarrow F_0(w)$ has odd symmetry about $\pi/2$.

Since $F_0(w) = V(w) + 1$.

So product filter $F_0(z)$ is a half band filter.
Product Filters (cont.)

Nonzero time component $V(\omega)$

Resulting Product Filter $F(\omega)$ with double zeros on unit circle
Relationship Between Product Filter and Analysis Filter $H_0(z)$

\[ F_0(z) = H_0(z) H_0(z^{-1}) \]

implies

\[ F_0(\omega) = F_0(z) \bigg|_{z=e^{j\omega}} = H_0(z) H_0(z^{-1}) \bigg|_{z=e^{j\omega}} \]

\[ \Rightarrow F_0(\omega) = H_0(\omega) H_0^*(\omega) = |H_0(\omega)|^2. \]

Since $|H_0(\omega)| = |H^*(\omega)|$, this means that $H_0(z)$ is also a half-band filter whose magnitude response is the square root of that of product filter $F_0(z)$. So product filter has similar shape as analysis filter. There is no simple relationship between their transition widths, though.
Designing a CQF:

First design $F_0(z)$. Then obtain $H_0(z)$.

Any filter of the form

$$f_0(n) = w(n) \frac{\sin(\pi n/2)}{\pi n}$$

will meet the above criterion, where $w(n)$ is a window function (Kaiser, etc.) i.e.

$$F_1(z) = F_0(-z)$$

$f_0(n)$ is a low pass filter with cutoff frequency at $f_c = 1/4$ (half-band filter)

Use standard window method for designing FIR filters.
CQF Design (cont.)

How to ensure \( F_0(Z) = H_0(Z) H_0^*(Z) \)?

For every zero of \( F_0(Z) \) at \( r e^{j\phi} \), there must be a corresponding zero at \( \frac{1}{r} e^{j\phi} \)

\[
F_0(Z) = K \prod_{m=1}^{N} (Z-Z_m)(Z^{-1}-Z_m) \quad K = \text{real constant}
\]

\( \Rightarrow \) any FIR filter with real, symmetric coeff (zero phase).

If zeros on unit circle, must lie in complex pairs or at \( Z = \pm 1 \).
CQF Design (cont.)

1. Start with \( f_0(n) = \omega(n) \frac{\sin(\pi n/2)}{\pi n} \) length of \( f_0(n) = 2L-1 \)

2. Transform \( f'_0(n) = a f_0(n) + b \quad -L+1 \leq n \leq L-1 \)

   \( a, b \) are constants.

\( L = \text{length of } h_0(n). \)

It can be shown that

\[
a = \frac{1}{1 + 10^{-A_p/20}}
\]

\[
b = 10^{-A_p/20}
\]

\( b \) moves zeros off of unit circle.

\( a \) normalizes passband magnitude response to 1.
CQF Design (cont.)

Given $A_a$, determine $A_p$ and then design $F_0(z)$.

Once $F_0(z)$ is designed,

$$F_i(z) = F_0(-z) = K \prod_{m=1}^{N} (z + z_m)(z^{-1} + z_m)$$

$$H_0(z) = \left[ \sqrt{K} \prod_{m=1}^{N} (z - z_m) \right] z^{-N}$$

$$H_1(z) = \sqrt{K} \prod_{m=1}^{N} (z^{-1} + z_m)$$

$$G_0(z) = \sqrt{K} \prod_{m=1}^{N} (z^{-1} - z_m)$$

$$G_1(z) = -\left[ \sqrt{K} \prod_{m=1}^{N} (z + z_m) \right] z^{-N}$$

(From Slides 22-23)

where $K$ is a constant.

Perfect Reconstruction and are good analysis filters.

But $H_0(z)$ is NOT linear phase.

(can make $H_0(z)$ approximate linear phase)
Product Filter Transformation: Matlab Example

```matlab
M=9; Fc=0.25;
hd = zeros(1,M);
for n=0:M-1
  if (n == (M-1)/2) % watch out in case M is odd
    hd(n+1) = sin(2*pi*Fc * (n - (M-1)/2)) / (pi * (n - (M-1)/2));
  else
    hd(n+1) = 2*Fc;
  end
end

% create Blackman window function
m = 0:1:M-1;
w = 0.42 - 0.5.*cos(2*pi*(m)/(M-1)) + 0.08 .* cos(4*pi*(m)/(M-1));

% create filter impulse response
h = hd .* w;

for i=0.2:-0.04:0.01
  hh = h + i;
zplane(hh,[1]);
title('Pole-zero plot in Z-Plane');
pause
[H,w] = freqz(hh,[1],376);
plot(w/(2*pi),20*log10(abs(H)));
grid;
title('Product Filter Transformation for different b');
xlabel('Frequency (Hz)');
ylabel('Magnitude (dB)');
pause
end
```
Effect of Transformation $f'_0(n) = f_0(n) + b$

Note the zeros are always in reciprocal pairs

$b = 0.2$

$b = 0.16$
Effect of Transformation $f'_0(n) = f_0(n) + b$

$b=0.12$

$b=0.08$
Effect of Transformation $f_0'(n) = f_0(n) + b$

As $b$ decreases, the zeros move away from the unit circle.

In the above example, the magnitude responses have not been normalized (via parameter $a$).
Summary of CQF Characteristics

Given a product filter, many analysis filters can be derived that have the same magnitude response: for each reciprocal zero pair, either zero could be used in the analysis filter.

Usually, among all the possible analysis filters, the one with the most “linear phase” response is often chosen and nearly linear phase can sometimes be obtained.

In summary, CQF design uses straightforward FIR window design techniques to obtain good filters with narrow transition widths AND guarantees perfect reconstruction. However, the CQF but does not guarantee exact linear phase analysis filters.